E2 212 Matrix Theory
Mid Term 1

Time: 90 Minutes **Date: 10/09/2026**

Instructions:

- Be concise and precise in your answers.
- Avoid unnecessary elaboration, only the required steps and final results will be evaluated.
- Each question has **two pages** of space allotted — use as much as you need.
- No extra sheets will be provided.

INDIAN INSTITUTE OF SCIENCE
भारतीय विज्ञान संस्थान

Question 1 4 points

A matrix $A \in \mathbb{C}^{n\times n}$ is said to be normal if

$$AA^H = A^H A.$$

Prove that if A is normal, then

$$\mathcal{R}(A) \perp \mathcal{N}(A).$$

Solution.

Step 1: $\mathcal{N}(A) = \mathcal{N}(A^H)$ when A is normal.

For any $x \in \mathbb{C}^n$,

$$\|Ax\|_2^2 = (Ax)^H(Ax) = x^H A^H Ax = x^H AA^H x = (A^H x)^H(A^H x) = \|A^H x\|_2^2,$$

where the middle equality uses normality, $A^H A = AA^H$.

Hence $\|Ax\|_2 = \|A^H x\|_2$ for every x, so

$$Ax = 0 \iff A^H x = 0,$$

i.e. $\mathcal{N}(A) = \mathcal{N}(A^H)$.

Step 2: General fact — $\mathcal{R}(A) \perp \mathcal{N}(A^H)$.

For any $A \in \mathbb{C}^{n\times n}$ and $y \in \mathbb{C}^n$:

$$y \in \mathcal{R}(A), z \in \mathcal{N}(A^H) \iff Ax = y \ \forall x,\ A^H z = 0,\ z \in \mathcal{N}(A^H) \iff x^H A^H = y^H \ \forall x, A^H z = 0,\ z \in \mathcal{N}(A^H)$$

$$\iff x^H(A^H z) = y^H z = 0 \iff \mathcal{R}(A) \perp \mathcal{N}(A^H).$$

Step 3: Combine.

From Steps 1 and 2,

$$\mathcal{R}(A) \perp \mathcal{N}(A^H) = \mathcal{N}(A).$$

Therefore $\mathcal{R}(A) \perp \mathcal{N}(A)$.

INDIAN INSTITUTE OF SCIENCE
भारतीय विज्ञान संस्थान

Question 2 4 points

Let $x \in \mathbb{R}^n$. Show that

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\,\|x\|_2,$$

and

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\,\|x\|_\infty,$$

When is equality attained in each of these inequalities?

Solution.

(i) $\|x\|_2 \le \|x\|_1$**.**

$$\|x\|_1^2 = \Big(\sum_{i=1}^n |x_i|\Big)^2 = \sum_i x_i^2 + \sum_{i\neq j} |x_i||x_j| \ge \sum_i x_i^2 = \|x\|_2^2.$$

Equality iff $x_i x_j = 0$ for all $i \neq j$ $\iff$ **at most one component of** x **is nonzero**.

(ii) $\|x\|_1 \le \sqrt{n}\|x\|_2$**.** By Cauchy–Schwarz on $(1,\dots,1)$ and $(|x_1|,\dots,|x_n|)$:

$$\|x\|_1 = \sum_i 1\cdot|x_i| \le \Big(\sum_i 1^2\Big)^{1/2}\Big(\sum_i |x_i|^2\Big)^{1/2} = \sqrt{n}\,\|x\|_2.$$

Equality $\iff$ **all** $|x_i|$ **are equal**.

(iii) $\|x\|_\infty \le \|x\|_2$**.** With $k = \arg\max_i |x_i|$,

$$\|x\|_\infty^2 = |x_k|^2 \le \sum_i |x_i|^2 = \|x\|_2^2.$$

Equality $\iff$ **at most one component of** x **is nonzero**.

(iv) $\|x\|_2 \le \sqrt{n}\|x\|_\infty$**.**

$$\|x\|_2^2 = \sum_i |x_i|^2 \le \sum_i \|x\|_\infty^2 = n\|x\|_\infty^2 \implies \|x\|_2 \le \sqrt{n}\|x\|_\infty.$$

Equality $\iff$ **all** $|x_i|$ **are equal** (and equal to the max).

INDIAN INSTITUTE OF SCIENCE
भारतीय विज्ञान संस्थान

Question 3

Let $x \in \mathbb{R}^n$, and let $|x_{i_1}| \geq |x_{i_2}| \geq \cdots \geq |x_{i_n}|$ denote the entries of x sorted in decreasing order of magnitude. Define the k-norm

$$\|x\|_{[k]} = |x_{i_1}| + |x_{i_2}| + \cdots + |x_{i_k}|, \qquad 1 \leq k \leq n.$$

(a) Prove that $\|\cdot\|_{[k]}$ is a norm on $\mathbb{R}^n$.

(b) Does $\|\cdot\|_{[k]}$ behave like the ℓ_1 norm or the ℓ_∞ norm? If yes, explain how; if not, explain why not.

Solution.

(a) Key observation. $\|x\|_{[k]}$ can equivalently be written as a maximum over all size-k index subsets:

$$\|x\|_{[k]} = \max_{\substack{S \subseteq \{1,\ldots,n\} \\ |S|=k}} \sum_{i \in S} |x_i|,$$

since choosing S to be the indices of the k largest $|x_i|$'s is exactly what maximizes the sum $\sum_{i \in S} |x_i|$ over all size-k subsets. This reformulation makes the triangle inequality easy to prove.

(i) Non-negativity. Each term $|x_{i_j}| \geq 0$, so $\|x\|_{[k]} \geq 0$.

(ii) Definiteness. If $x = 0$, all $|x_i| = 0$, so $\|x\|_{[k]} = 0$. Conversely, if $\|x\|_{[k]} = 0$, then since $|x_{i_1}|$ is the *largest* of all $|x_i|$ and appears as a non-negative term in this sum, $|x_{i_1}| = 0$; since $|x_{i_1}| \geq |x_i|$ for all i, every $|x_i| = 0$, i.e. $x = 0$.

(iii) Homogeneity. For a scalar α, $|\alpha x_i| = |\alpha||x_i|$ for every i; multiplying by the non-negative scalar $|\alpha|$ preserves the relative ordering, so the top-k index set is unchanged. Hence

$$\|\alpha x\|_{[k]} = \sum_{j=1}^{k} |\alpha x_{i_j}| = |\alpha| \sum_{j=1}^{k} |x_{i_j}| = |\alpha| \, \|x\|_{[k]}.$$

(iv) Triangle inequality. Let $S^\star$ be a size-k set achieving the max for $x + y$:

$$\|x+y\|_{[k]} = \sum_{i \in S^\star} |x_i + y_i| \leq \sum_{i \in S^\star} |x_i| + \sum_{i \in S^\star} |y_i| \leq \|x\|_{[k]} + \|y\|_{[k]},$$

where the first step is the ordinary triangle inequality termwise, and the second uses that any particular size-k sum is at most the maximum size-k sum.

All four axioms hold, so $\|\cdot\|_{[k]}$ is a norm on $\mathbb{R}^n$.

(b) Relationship to ℓ_1 and ℓ_∞.

$\|\cdot\|_{[k]}$ is neither the ℓ_1 norm nor the ℓ_∞ norm in general — rather, it *interpolates* between them, coinciding with each only at the two extreme values of k.

The two extreme cases.

- $k = 1$**:** $\|x\|_{[1]} = |x_{i_1}| = \max_i |x_i| = \|x\|_\infty$. Here the k-norm reduces *exactly* to ℓ_∞.
- $k = n$**:** $\|x\|_{[n]} = \sum_{j=1}^{n} |x_{i_j}| = \sum_i |x_i| = \|x\|_1$. Here it reduces *exactly* to ℓ_1.

Question 4 4 points

Let R be an inner product space. Prove that for all $x, y \in \mathbb{R}^n$,

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|,$$

where the norm is defined by

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Show that equality holds if and only if x and y are linearly dependent. In particular, for $x \neq 0$, equality holds when

$$y = \alpha x, \qquad \alpha = \frac{\langle x, y\rangle}{\langle x, x\rangle}.$$

Solution.

If $x = 0$, both sides equal 0 and x, y are trivially linearly dependent, so assume $x \neq 0$. Set

$$\alpha = \frac{\langle x, y\rangle}{\langle x, x\rangle} = \frac{\langle x, y\rangle}{\|x\|^2},$$

Step 1: Expand $0 \leq \|\alpha x - y\|^2$.
Since the norm is non-negative,

$$0 \leq \|\alpha x - y\|^2 = \langle \alpha x - y, \ \alpha x - y\rangle = \bar{\alpha}\langle x, \alpha x - y\rangle - \langle y, \alpha x - y\rangle.$$

The first term vanishes by the orthogonality just established, so

$$0 \leq -\langle y, \alpha x - y\rangle = \langle y, y\rangle - \alpha\langle y, x\rangle = \|y\|^2 - \alpha\,\overline{\langle x, y\rangle}.$$

Step 2: Substitute α.
Using $\alpha = \dfrac{\langle x, y\rangle}{\|x\|^2}$ and $\langle x, y\rangle\,\overline{\langle x, y\rangle} = |\langle x, y\rangle|^2$:

$$0 \leq \|y\|^2 - \frac{\langle x, y\rangle}{\|x\|^2}\,\overline{\langle x, y\rangle} = \|y\|^2 - \frac{|\langle x, y\rangle|^2}{\|x\|^2}.$$

Step 3: Clear the denominator.
Since $\|x\|^2 > 0$ (as $x \neq 0$), multiplying through by $\|x\|^2$ preserves the inequality:

$$0 \leq \|x\|^2\|y\|^2 - |\langle x, y\rangle|^2 \quad \Longrightarrow \quad |\langle x, y\rangle|^2 \leq \|x\|^2\|y\|^2.$$

Taking square roots (both sides non-negative),

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|.$$

Equality condition.
Tracing the argument backward, equality in the final inequality holds if and only if equality holds in Step 1, i.e.

$$0 = \|\alpha x - y\|^2 \iff \alpha x - y = 0 \iff y = \alpha x,$$

i.e. y is a scalar multiple of x — precisely the statement that $\{x, y\}$ is linearly dependent.

Conversely, if x, y are linearly dependent with $x \neq 0$, then $y = \alpha x$ for some scalar α, and direct substitution gives

$$|\langle x, y\rangle| = |\alpha| \, \langle x, x\rangle = |\alpha| \, \|x\|^2 = \|x\| \cdot \big(|\alpha| \, \|x\|\big) = \|x\| \, \|\alpha x\| = \|x\| \, \|y\|,$$

confirming equality holds precisely when $y = \alpha x$ with $\alpha = \dfrac{\langle x, y\rangle}{\langle x, x\rangle}$.

Question 5 5 points

(a) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, and let $\|\cdot\|^D$ denote the corresponding dual norm. Let $\mathbf{e}_i$ be a standard basis vector.

Show that

$$\|\mathbf{e}_i\| \, \|\mathbf{e}_i\|^D \geq 1.$$

(b) Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two given norms on $\mathbb{C}^n$, and suppose that there exists some $C > 0$ such that

$$\|x\|_\alpha \leq C\|x\|_\beta, \qquad \text{for all } x \in \mathbb{C}^n.$$

Explain why

$$\|x\|_\beta^D \leq C\|x\|_\alpha^D, \qquad \text{for all } x \in \mathbb{C}^n.$$

Solution.

(a) By definition of the dual norm,

$$\|y\|^D = \sup_{x \neq 0} \frac{|\langle x, y\rangle|}{\|x\|},$$

which is equivalent to

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|^D \qquad \text{for all } x, y.$$

Setting $x = y = \mathbf{e}_i$ and using $\langle \mathbf{e}_i, \mathbf{e}_i\rangle = 1$:

$$1 = |\langle \mathbf{e}_i, \mathbf{e}_i\rangle| \leq \|\mathbf{e}_i\| \, \|\mathbf{e}_i\|^D.$$

This proves $\|\mathbf{e}_i\| \, \|\mathbf{e}_i\|^D \geq 1$.

(b) Fix $y \in \mathbb{C}^n$. By definition,

$$\|y\|_\beta^D = \sup_{x \neq 0} \frac{|\langle x, y\rangle|}{\|x\|_\beta}.$$

From the hypothesis $\|x\|_\alpha \leq C\|x\|_\beta$, we get, for every $x \neq 0$,

$$\|x\|_\beta \geq \frac{1}{C}\|x\|_\alpha \quad \Longrightarrow \quad \frac{1}{\|x\|_\beta} \leq \frac{C}{\|x\|_\alpha}.$$

Therefore, for every $x \neq 0$,

$$\frac{|\langle x, y\rangle|}{\|x\|_\beta} \;\leq\; C\,\frac{|\langle x, y\rangle|}{\|x\|_\alpha} \;\leq\; C\,\|y\|_\alpha^D,$$

where the last step uses the definition of $\|y\|_\alpha^D$ as a supremum.
Taking the supremum over $x \neq 0$ on the left-hand side gives

$$\|y\|_\beta^D \leq C\,\|y\|_\alpha^D.$$

Since y was arbitrary, this holds for all $x \in \mathbb{C}^n$ (relabeling), as required.